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Extremal and nonextremal Kerr/CFT correspondences

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Abstract

I rederive the Kerr/CFT correspondence without first taking the near-horizon extremal Kerr limit. This method extends easily to nonextremal black holes, for which the temperature and central charge behave poorly at the horizon but the entropy remains finite. A computation yields one-half of the standard Bekenstein-Hawking entropy, with hints that the other half may be related to a conformal field theory at the inner horizon. I then present an alternative approach, based on a stretched Killing horizon, in which the full entropy is obtained and the temperature and central charge remain well-behaved even in the nonextremal case.

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The idea that black hole thermodynamics might be explained by a near-horizon conformal symmetry [1–3] has gained new currency with the recent discovery of an extremal Kerr/CFT duality [4]. A near-horizon conformal field theory is an attractive idea: it might help explain the "universality" of black hole entropy, and suggests a picture of the relevant degrees of freedom as Goldstone-like excitations arising from the conformal anomaly [5]. For the special case of an extremal Kerr black hole, the correspondence found in [4] has additional attractive features, including a particularly simple central charge and perhaps a microscopic string theory realization [6]. But although some progress has been made in extending this correspondence to near-extremal black holes [7, 8], the application to black holes far from extremality remains obscure.

The usual approach to the extremal Kerr/CFT correspondence begins by approximating the extremal Kerr metric near the horizon as the "near-horizon extremal Kerr," or NHEK, metric of Bardeen and Horowitz [9]. This form of the metric makes the near-horizon AdS_2 structure explicit, but does not generalize easily to the nonextremal case, where the near-horizon geometry is Rindler rather than anti-de Sitter. But the NHEK limit, while convenient, should not be necessary: the horizon symmetry must surely be present in the full extremal Kerr metric, although perhaps better hidden.

In the first section of this paper, I rederive the extremal Kerr/CFT correspondence without first going to the NHEK limit, using instead a stretched horizon formalism. I then extend the calculation to an arbitrary stationary (3+1)-dimensional black hole, without requiring (near)-extremality. For nonextremal black holes, the temperature and central charge computed in this manner behave badly at the horizon, but the combination that gives the entropy remains finite, yielding half the standard Bekenstein-Hawking entropy. The factor of one-half can be traced to the presence of a single zero in the lapse function for a nonextremal black hole, in contrast to the double zero in the extremal case. Since the extremal double zero comes from the merger of the inner and outer horizons, this suggests that the "missing" entropy may be related to the inner horizon.

The bad behavior of the temperature and central charge at the horizon make this approach somewhat unsatisfying, however. I therefore present an alternative near-horizon conformal field theory, living on a stretched Killing horizon. This model, which is closely related to that of [3], has a finite temperature and central charge even as the stretched horizon approaches the true event horizon, while the singular behavior is shifted to a physical infinite blue shift at the horizon. The Cardy formula, in both canonical and microcanonical form, now yields the correct Bekenstein-Hawking entropy.

The covariant phase space approach to the central charge—first applied to black holes in [3], corrected and significantly extended in [10, 11], and perfected in [12, 13]—provides an elegant geometric description of the properties of a boundary conformal field theory. Unfortunately, though, this formalism is a bit less transparent than one might hope for. I therefore use the more straightforward canonical ADM approach pioneered by Brown and Henneaux [14], which I briefly review in an appendix.

1 Extremal Kerr/CFT without NHEK

Let us begin with the extremal Kerr metric in Boyer-Lindquist coordinates, written in ADM form as [15]

$$\begin{aligned} ds^2 &= -N^2 dt^2 + q_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \\ &= -N^2 dt^2 + \frac{\Sigma}{\Delta} dr^2 + \frac{A \sin^2\theta}{\Sigma} (d\varphi + N^\varphi dt)^2 + \Sigma\, d\theta^2 \ , \end{aligned} \tag{1.1}$$

where q_{ij} denotes the spatial metric on a constant time slice, and

$$\begin{aligned} &\Delta = (r - r_+)^2, \qquad \Sigma = r^2 + {r_+}^2 \cos^2\theta, \qquad A = (r^2 + {r_+}^2)^2 - (r - r_+)^2 {r_+}^2 \sin^2\theta, \\ &N = \sqrt{\frac{\Sigma\Delta}{A}}, \qquad N^\varphi = \frac{2{r_+}^2 r}{A} \ . \end{aligned} \tag{1.2}$$

The only nonvanishing component of the canonical momentum is

$$\pi^{r\varphi} = \frac{\sqrt{q}}{2N} q^{rr} \partial_r N^\varphi \ . \tag{1.3}$$

Near the horizon, the shift vector N^φ can be expanded as $N^\varphi \approx -\Omega_H + \varepsilon$, where Ω_H is the horizon angular velocity and the small parameter ε is given by

$$\varepsilon = (r - r_+)\partial_r N^\varphi \big|_{r=r_+} = -\frac{(r - r_+)}{2{r_+}^2} \ . \tag{1.4}$$

Under a diffeomorphism generated by a vector field ξ^μ, the metric transforms as

$$\begin{aligned} \delta_\xi N \ &= \bar\partial_t \xi^\perp + \hat\xi^i \partial_i N \\ \delta_\xi N^i &= \bar\partial_t \hat\xi^i - N q^{ij} \partial_j \xi^\perp + q^{ik} \partial_k N \xi^\perp + \hat\xi^j \partial_j N^i \\ \delta_\xi q_{ij} \ &= q_{ik}\left(\partial_j \hat\xi^k - \frac{\partial_j N^k}{N}\xi^\perp\right) + q_{jk}\left(\partial_i \hat\xi^k - \frac{\partial_i N^k}{N}\xi^\perp\right) + \frac{1}{N}\xi^\perp \bar\partial_t q_{ij} + \hat\xi^k \partial_k q_{ij} \ , \end{aligned} \tag{1.5}$$

where

$$\bar\partial_t = \partial_t - N^i \partial_i = \partial_t + \Omega_H \partial_\varphi - \varepsilon \partial_\varphi \tag{1.6}$$

is a convective derivative, and the quantities $(\xi^\perp, \hat\xi^i)$ are the "surface deformation parameters" (A.6) that appear in the ADM Hamiltonian [14,16]. Note that for any function of the NHEK angular coordinate $\phi = \varphi - \Omega_H t$,

$$\bar\partial_t f = -\varepsilon \partial_\varphi f \ . \tag{1.7}$$

For extremal black holes, the horizon $\mathcal{H}$ is infinitely far from any stationary observer, in the sense that the proper distance from r_+ to any point $r > r_+$ is infinite. The NHEK approach therefore treats boundary conditions at the horizon as asymptotic fall-off conditions. In Boyer-Lindquist coordinates, on the other hand, the coordinate distance to the horizon is

finite, and we must instead impose boundary conditions at $r = r_+$. As usual, it is difficult to do this precisely at the horizon. In the ADM coordinates (1.1), N goes to zero at $\mathcal{H}$, and the metric becomes singular. One could, of course, choose coordinates that are well-behaved at the horizon, but the real problem is more general: the horizon is a null surface, and the presence of second class constraints makes boundary conditions at such a surface extremely complicated. I will therefore impose boundary conditions at a "stretched horizon" $\mathcal{H}_s$, and then take the limit as $\mathcal{H}_s$ approaches the true horizon $\mathcal{H}$.

As we shall later in this paper, there is more than one way to stretch a horizon. It turns out the NHEK boundary conditions of [4] correspond to fixing the angular velocity N^φ, or equivalently the parameter ε, thus determining a surface that rotates at a constant angular velocity $\Omega_s = \Omega_{\scriptscriptstyle H} - \varepsilon$ that differs slightly from $\Omega_{\scriptscriptstyle H}$. Indeed, if ε is fixed, it is easy to see from (1.5) and (1.7) that $\delta_\xi N^r$ will be small as long as ξ^r is a function of $\varphi - \Omega_{\scriptscriptstyle H} t$. Then

$$\delta_\xi N^\varphi = \bar{\partial}_t \hat{\xi}^\varphi - N^2 q^{\varphi\varphi}\partial_\varphi \xi^t + \hat{\xi}^r \partial_r N^\varphi = 0 \tag{1.8}$$

has a solution

$$\hat{\xi}^r = (r - r_+)\partial_\varphi \hat{\xi}^\varphi, \qquad \xi^t = \mathcal{O}(r - r_+) \ . \tag{1.9}$$

As in [4], these transformations allow $\mathcal{O}(1)$ changes in q_{rr} and $q_{\varphi\varphi}$, but still lead to a well-behaved variational principle; in particular, the variations of the conjugate variables π^{rr} and $\pi^{\varphi\varphi}$ vanish as $r \to r_+$. In fact, the group of diffeomorphisms (1.9) is equivalent to the asymptotic symmetry group of the NHEK metric found in [4]: not only is the algebra the same, but the vector fields themselves, when transformed to NHEK coordinates, match those of [4] near the horizon.

We can now exploit the Cardy formula to determine the density of states. This formula is most often seen in its microcanonical form [17,18], and I discuss this form in Appendix B, but for our purposes the canonical version (see, for instance, section 8 of [19]) is more convenient. For this, we need both the temperature T and the central charge c. The derivation of the temperature in [4] did not involve the NHEK limit, and we can use that result directly:

$$T = \frac{1}{2\pi} \ . \tag{1.10}$$

I will return to this result in the next section, where a related but slightly different derivation is available for the nonextremal case.

To determine the central charge, we can turn to the expression (A.10) of Appendix A, which gives the general central term of the surface deformation algebra in the canonical formalism. For the group of deformations (1.9), it is easy to check that the only term that remains nonzero at the horizon is

$$\begin{aligned} K[\xi,\eta] &= -\frac{1}{8\pi G}\int_{\partial\Sigma} d^2x \frac{\sqrt{\sigma}}{\sqrt{q}}\, n^k(\hat{\eta}_k \pi^{mn} D_m \hat{\xi}_n - \hat{\xi}_k \pi^{mn} D_m \hat{\eta}_n) \\ &= -\frac{1}{8\pi G}\int_{\mathcal{H}_s} d^2x \sqrt{\sigma}\, n^r q_{rr} \hat{\xi}^r \left(\frac{1}{2N} q^{rr}\partial_r N^\varphi\right) q_{rr}\partial_\varphi \hat{\eta}^r - (\hat{\xi} \leftrightarrow \hat{\eta}) \\ &= -\frac{1}{16\pi G}\int_{\mathcal{H}_s} d^2x \sqrt{\sigma}\, \frac{n_r}{N}\partial_r N^\varphi (r - r_+)^2 \left(\partial_\varphi \hat{\xi}^\varphi \partial_\varphi^2 \hat{\eta}^\varphi - \partial_\varphi \hat{\eta}^\varphi \partial_\varphi^2 \hat{\xi}^\varphi\right) \ . \end{aligned} \tag{1.11}$$

But near the horizon,

$$\frac{n_r}{N} = \frac{\sqrt{A}}{\Delta} \approx \frac{2{r_+}^2}{(r-r_+)^2}, \qquad \partial_r N^\varphi \approx -\frac{1}{2{r_+}^2}, \tag{1.12}$$

so

$$\begin{aligned} K[\xi,\eta] &= \frac{1}{16\pi G}\int_{\mathcal{H}_s} d^2x\sqrt{\sigma}\,\left(\partial_\varphi\hat{\xi}^\varphi\partial_\varphi^2\hat{\eta}^\varphi - \partial_\varphi\hat{\eta}^\varphi\partial_\varphi^2\hat{\xi}^\varphi\right) \\ &= \frac{1}{16\pi G}\frac{\mathcal{A}}{2\pi}\int d\varphi\,\left(\partial_\varphi\hat{\xi}^\varphi\partial_\varphi^2\hat{\eta}^\varphi - \partial_\varphi\hat{\eta}^\varphi\partial_\varphi^2\hat{\xi}^\varphi\right)\ , \end{aligned} \tag{1.13}$$

where $\mathcal{A} = \int d^2x\sqrt{\sigma} = 8\pi{r_+}^2$ is the horizon area and I have used the fact that the metric near the horizon is independent of φ.

We can recognize (1.13) as the central term in a Virasoro algebra [20] with central charge

$$c = 48\pi\frac{1}{16\pi G}\frac{\mathcal{A}}{2\pi} = \frac{3\mathcal{A}}{2\pi G} = 12J\ , \tag{1.14}$$

precisely matching the NHEK central charge of [4]. It is amusing to note that this quantity also matches the old near-horizon results of [2,3]. The canonical version of the Cardy formula then yields an entropy

$$S = \frac{\pi^2}{3}cT = 2\pi J = \frac{\mathcal{A}}{4G}\ , \tag{1.15}$$

the standard Bekenstein-Hawking entropy.

2 Nonextremal black holes

Extremality played a very small role in the preceding section, and it is straightforward to generalize the construction to the nonextremal Kerr black hole. In fact, we can go farther, and consider an arbitrary stationary black hole. Near the horizon, the metric of a stationary nonextremal (3+1)-dimensional black hole can always be written in the ADM form

$$ds^2 = -N^2dt^2 + d\rho^2 + q_{\varphi\varphi}\left(d\varphi + N^\varphi dt\right)^2 + q_{zz}dz^2\ , \tag{2.1}$$

where ρ is the proper distance from the horizon and z is, for example, $\cos\theta$. Independent of any field equations, the requirement that the curvature be finite at the horizon sharply restricts the behavior of the metric: it must have an expansion of the form [21]

$$\begin{aligned} &N = \kappa_{\scriptscriptstyle H}\rho + \frac{1}{3!}\kappa_2(z)\rho^3 + \dots && q_{\varphi\varphi} = [q_H]_{\varphi\varphi}(z) + \frac{1}{2}[q_2]_{\varphi\varphi}(z)\rho^2 + \dots \\ &N^\varphi = -\Omega_{\scriptscriptstyle H} - \frac{1}{2}\omega_2(z)\rho^2 + \dots && q_{zz} = [q_H]_{zz}(z) + \frac{1}{2}[q_2]_{zz}(z)(\rho^2) + \dots\ , \end{aligned} \tag{2.2}$$

where the surface gravity $\kappa_{_H}$ and the horizon angular velocity $\Omega_{_H}$ are constants. The only nonvanishing component of the canonical momentum π^{ij} at the horizon is

$$\pi^{\rho\varphi} = -\frac{\omega_2}{2\kappa_{_H}}\sqrt{q} + \mathcal{O}(\rho^2) \ . \tag{2.3}$$

As in the preceding section, we impose boundary conditions that $N^\varphi = -\Omega_{_H} + \varepsilon$ is fixed at the stretched horizon $\mathcal{H}_s$. Equation (1.8) now has a solution*

$$\hat{\xi}^\rho = -\frac{\rho}{2}\partial_\varphi\hat{\xi}^\varphi, \qquad \xi^t = \mathcal{O}(\rho) \ , \tag{2.4}$$

with $\hat{\xi}^\varphi$ a function of the corotating coordinate $\varphi - \Omega_{_H}t$. As in the extremal case, these diffeomorphisms preserve a sufficient set of boundary data on the stretched horizon.

Again as in the preceding section, we can use (A.10) to determine the central charge. The result is now

$$\begin{aligned} K[\xi,\eta] &= -\frac{1}{8\pi G}\int_{\partial\Sigma} d^2x\frac{\sqrt{\sigma}}{\sqrt{q}}\, n^k(\hat{\eta}_k\pi^{mn}D_m\hat{\xi}_n - \hat{\xi}_k\pi^{mn}D_m\hat{\eta}_n) \\ &= \frac{\varepsilon}{32\pi G\kappa_{_H}}\frac{\mathcal{A}}{2\pi}\int d\varphi\,(\partial_\varphi\hat{\xi}^\varphi\partial_\varphi^2\hat{\eta}^\varphi - \partial_\varphi\hat{\eta}^\varphi\partial_\varphi^2\hat{\xi}^\varphi) \ . \end{aligned} \tag{2.5}$$

This is again the central term of a Virasoro algebra, with central charge

$$c = \frac{3\mathcal{A}}{2\pi G}\frac{\varepsilon}{2\kappa_{_H}} \ . \tag{2.6}$$

To determine the appropriate temperature to use in the Cardy formula, we can adapt the arguments of [4]. First, as usual, the Hawking temperature is

$$T_H = \frac{\kappa_{_H}}{2\pi} \ . \tag{2.7}$$

As noted in [4], however, this is not the relevant temperature for our conformal algebra. The Frolov-Thorne vacuum is annihilated by modes with a coordinate dependence $e^{-i\omega t+im\varphi}$. On our stretched horizon, on the other hand, the relevant angular coordinate is $\tilde{\varphi} = \varphi - (\Omega_{_H}+\varepsilon)t$. The Frolov-Thorne modes thus become

$$\Phi_{m\omega} \sim e^{-i\tilde{\omega}t+im\tilde{\varphi}} \quad \text{with} \ \ \tilde{\omega} = \omega - m(\Omega_{_H}+\varepsilon), \tag{2.8}$$

with a Boltzmann factor $e^{-\beta(\omega-m\Omega_H)} = e^{-\beta(\tilde{\omega}-m\varepsilon)}$ The relevant temperature for $\tilde{\varphi}$ modes is thus

$$T = \frac{T_H}{\varepsilon} = \frac{\kappa_{_H}}{2\pi\varepsilon} \ . \tag{2.9}$$

*The factor of 1/2 difference between (2.4) and (1.9) is a coordinate artifact, coming from the fact that $\rho \sim (r - r_+)^{1/2}$.

The central charge (2.6) and temperature (2.9) are poorly behaved in the horizon limit $\varepsilon \to 0$. The entropy, however, is not. Indeed, inserting the central charge and temperature into the canonical Cardy formula, we obtain

$$S = \frac{\pi^2}{3} cT = \frac{\mathcal{A}}{8G} \ , \tag{2.10}$$

one-half of the Bekenstein-Hawking entropy.

Mathematically, the missing factor of two can be traced back to the fact that N^2 has a double zero at the horizon for an extremal black hole, but only a single zero for a nonextremal black hole. Indeed, for the nonextremal version of (1.11), the integrand contains a factor of

$$\frac{n_r}{N}(r - r_+) = \frac{n_r}{2\partial_r N} = \frac{1}{2\kappa_{{}_H}} \ , \tag{2.11}$$

where the factor of two comes from the fact that $N \sim \sqrt{r - r_+}$. For the extremal black hole, $N \sim r - r_+$, and this factor is not present.

This observation suggests that the missing entropy could come from a conformal field theory at the inner horizon. This is at least technically true, in the sense that the double zero of the extremal lapse function comes from the merger of single zeros at the inner and outer horizons. One might worry, though, both about the instability of the inner horizon [22] and about the physical question of whether thermodynamic properties of a black hole can depend on a region that is causally disconnected from the exterior.

A possible alternative would be to find a second Virasoro algebra at the (outer) stretched horizon $\mathcal{H}_s$, describing a second set of modes. While I cannot rule out the existence of such an algebra, I have not succeeded in finding one. In particular, the requirement of two *commuting* Virasoro algebras is a particularly strong one. It may be, as Castro et al. have argued [23], that one must search for a hidden nongeometric symmetry to obtain the remaining modes.

3 More than one way to stretch a horizon

The approach of the preceding sections has reproduced the NHEK expression for the entropy of an extremal black hole. We have seen, however, that the nonextremal results are less convincing, both because of the missing factor of two in the entropy and because of the poor behavior of the temperature and the central charge at the horizon. It is therefore worth looking for an alternative conformal description.

As noted above, there is more than one way to stretch a horizon. The approach so far has been based on the fact that the horizon has a constant angular velocity $\Omega_{{}_H}$; our stretched horizon has been a surface with a slightly different angular velocity $\Omega_s = \Omega_{{}_H} - \varepsilon$. An alternative approach, closer to that of [2,3,11,24], is to note that the horizon of a stationary black hole† is a Killing horizon: that is, it admits a Killing vector $\chi^a = T^a + \Omega_{{}_H}\Phi^a$ that is null at $\mathcal{H}$ and is normal to $\mathcal{H}$. We cannot, of course, demand that the stretched horizon be

†The analysis is local, so one need not require that the spacetime be globally stationary. In fact, a general isolated horizon with a stationary neighborhood is a Killing horizon [25].

a Killing horizon, but we can "stretch" the Killing vector, by requiring that a new Killing vector

$$\bar{\chi}^a = T^a + \bar{\Omega}\Phi^a \tag{3.1}$$

be null at the stretched horizon. For the ADM metric (2.1), this means

$$\bar{\chi}^2 = 0 = -N^2 + q_{\varphi\varphi}(N^\varphi + \bar{\Omega})^2 = -N^2 + q_{\varphi\varphi}(\Omega_{\scriptscriptstyle H} - \bar{\Omega})^2 + \mathcal{O}(\rho^3)\ . \tag{3.2}$$

From the asymptotic behavior (2.2), it is evident that to lowest order we are fixing the proper distance ρ at $\mathcal{H}_s$, a procedure much closer to that of the usual "membrane paradigm" [26]. Note that the small parameter

$$\bar{\varepsilon} = \Omega_{\scriptscriptstyle H} - \bar{\Omega}\ , \tag{3.3}$$

which measures the "stretching" of the Killing vector $\bar{\chi}^a$, is of now order ρ. This contrasts with the parameter ε of section 1, which was of order ρ^2. It remains true, however, that as $\bar{\varepsilon} \to 0$, the stretched horizon approaches the true horizon. Note also that although $\bar{\Omega}$ characterizes the "stretched Killing vector," it is not the angular velocity of the stretched horizon. Rather, by (2.2), the stretched horizon has an angular velocity $-N^\varphi(\rho_s) = \Omega_{\scriptscriptstyle H} + \mathcal{O}(\bar{\varepsilon}^2)$.

We now require that the form (2.1) of the metric remain fixed at the stretched horizon—that is, that the lapse function N, the shift vector N^i, and the (absent) cross-term $q_{\rho\varphi}$ be unchanged to lowest order in ρ. Since we are treating the stretched horizon as a boundary, we can separately specify the surface deformation parameters $(\xi^\perp, \hat{\xi}^i)$ and their first normal derivatives $(\partial_\rho\hat{\xi}^i, \partial_\rho\xi^\perp)$ at $\mathcal{H}_s$. Let us consider parameters ξ^t of the form $\xi^t = \xi^t(\varphi - \bar{\Omega}t)$, where for the moment we leave $\bar{\Omega}$ arbitrary; it will later be fixed by imposing (3.2). From (1.5), we then find that at the stretched horizon,

$$\begin{aligned}
&\delta_\xi N = 0 \quad \Rightarrow \quad \hat{\xi}^\rho = -\bar{\varepsilon}\rho\partial_\varphi\xi^t = -\rho\bar{\partial}_t\xi^t \\
&\delta_\xi N^\rho = 0 \quad \Rightarrow \quad \rho\partial_\rho\xi^t = -\frac{\bar{\varepsilon}^2}{\kappa_{\scriptscriptstyle H}^2}\partial_\varphi{}^2\xi^t = -\frac{1}{\kappa_{\scriptscriptstyle H}^2}\bar{\partial}_t{}^2\xi^t \\
&\delta_\xi N^\varphi = 0 \quad \Rightarrow \quad \hat{\xi}^\varphi = \frac{\kappa_{\scriptscriptstyle H}^2\rho^2}{\bar{\varepsilon}}q^{\varphi\varphi}\xi^t = \bar{\varepsilon}\nu^2\xi^t \\
&\delta q_{\rho\varphi} = 0 \quad \Rightarrow \quad \rho\partial_\rho\hat{\xi}^\varphi = \bar{\varepsilon}\rho^2 q^{\varphi\varphi}\partial_\varphi{}^2\xi^t - \omega_2\rho^2\xi^t = \frac{\bar{\varepsilon}}{\kappa_{\scriptscriptstyle H}^2}\bar{\partial}_t{}^2\xi^t - \omega_2\rho^2\xi^t
\end{aligned} \tag{3.4}$$

where

$$\nu^2 = \frac{\kappa_{\scriptscriptstyle H}^2\rho^2}{q_{\varphi\varphi}\bar{\varepsilon}^2} = \frac{N^2}{q_{\varphi\varphi}(\Omega_{\scriptscriptstyle H} - \bar{\Omega})^2}\ . \tag{3.5}$$

We can also consistently require

$$\delta_\xi g_{\rho\rho} = 0 \quad \Rightarrow \quad \partial_\rho\hat{\xi}^\rho = 0 \tag{3.6}$$

at $\mathcal{H}_s$, although this condition will not be needed for what follows.

Under the surface deformation brackets (A.5), it is now straightforward to check that to lowest order,

$$\begin{aligned}
\{\xi,\eta\}^t_{SD} &= \bar{\varepsilon}(1+\nu^2)(\xi^t\partial_\varphi\eta^t - \eta^t\partial_\varphi\xi^t)\\
\{\xi,\eta\}^\rho_{SD} &= -\bar{\varepsilon}\rho\partial_\varphi\{\xi,\eta\}^t_{SD}\\
\{\xi,\eta\}^\varphi_{SD} &= \bar{\varepsilon}\nu^2\{\xi,\eta\}^t_{SD}\ .
\end{aligned} \tag{3.7}$$

These brackets form a Witt algebra—a Virasoro algebra with vanishing central charge—but with a nontrivial normalization:

$$\xi^t_n = \frac{1}{(1+\nu^2)\bar{\varepsilon}}\, e^{in(\varphi-\bar{\Omega}t)}\ . \tag{3.8}$$

They preserve the relations (3.4), providing a nontrivial consistency test. Note that the wave vector k_a determined by the exponent in (3.8) satisfies

$$k^2 = n^2 q^{\varphi\varphi}\left(1-\frac{1}{\nu^2}\right) + \mathcal{O}(\rho)\ , \tag{3.9}$$

and is null when $\nu^2 = 1$, that is, when (3.2) is obeyed.

The nonvanishing contributions to the central term (A.10) are now

$$\begin{aligned}
K[\xi,\eta] &= -\frac{1}{8\pi G}\int_{\partial\Sigma} d^2x\sqrt{\sigma}\, n^k\left[(D_i\hat{\xi}_k D^i\eta^\perp - D_i\hat{\xi}^i D_k\eta^\perp) - (\xi\leftrightarrow\eta)\right]\\
&= -\frac{1}{8\pi G}\int_{\mathcal{H}_s} d^2x\sqrt{\sigma}\left[(q^{\varphi\varphi}\partial_\varphi\hat{\xi}^\rho\partial_\varphi\eta^\perp - \partial_\varphi\hat{\xi}^\varphi\partial_\rho\eta^\perp) - (\xi\leftrightarrow\eta)\right]\\
&= -\frac{\nu^2\bar{\varepsilon}^3}{4\pi G\kappa_H}\frac{\mathcal{A}}{2\pi}\int d\varphi\ \left[\partial_\varphi\xi^t{\partial_\varphi}^2\eta^t - \partial_\varphi\eta^t{\partial_\varphi}^2\xi^t\right]\ .
\end{aligned} \tag{3.10}$$

Taking into account the normalization (3.8), we can recognize this as the central term for a Virasoro algebra with central charge

$$c = \frac{4\nu^2}{(1+\nu^2)^2}\frac{3\mathcal{A}}{2\pi G}\frac{\bar{\varepsilon}}{\kappa_H}\ . \tag{3.11}$$

We can now fix the angular velocity $\bar{\Omega}$ by demanding that the modes be invariant under the translation along the "stretched Killing vector" $\bar{\chi}$, or, equivalently, that $\bar{\chi}$ be invariant under boundary diffeomorphisms, i.e., $\mathcal{L}_\xi\bar{\chi} = 0$. Then by (3.2), $\nu^2 = 1$, and

$$c = \frac{3\mathcal{A}}{2\pi G}\frac{\bar{\varepsilon}}{\kappa_H}\ . \tag{3.12}$$

It is interesting to note that this choice of ν^2 gives the largest possible value of the central charge. Note also that by (3.9), this is the unique choice for which $k^2 = 0$. This is what one would expect from a two-dimensional conformal symmetry, since a null vector in Lorentzian signature translates to a holomorphic vector in Riemannian signature.

As in section 2, the temperature corresponding to the angular dependence (3.8) is

$$T = \frac{\kappa_H}{2\pi\bar{\varepsilon}} \ . \tag{3.13}$$

The canonical Cardy formula then yields

$$S = \frac{\pi^2}{3} cT = \frac{\mathcal{A}}{4G} \ , \tag{3.14}$$

as desired. I show in Appendix B that the same result can be obtained from the microcanonical version of the Cardy formula.

4 Taming the central charge

The approach of the preceding section has eliminated the "factor of two" problem. As in section 2, though, the temperature and the central charge for the nonextremal black hole are poorly behaved at the horizon. It may be, however, that this behavior really reflects something more physical, the infinite blue shift at the horizon relative to an observer outside the black hole.

To see this, let us replace the modes (3.8) with a slightly modified set,

$$\tilde{\xi}^t_n = \frac{1}{(1+\nu^2)}\, e^{in(\varphi - \bar{\Omega}t)/\bar{\varepsilon}} \ , \tag{4.1}$$

chosen so that $\bar{\partial}_t \tilde{\xi}^t_n = in\tilde{\xi}^t_n$. The frequencies of these modes blow up at the horizon, but this is essentially the ordinary blue shift. Indeed, a corotating observer (a ZAMO, or zero angular momentum observer) has a four-velocity $u^a = (T^a - N^\varphi \Phi^a)/N$, and by standard arguments will see a frequency $k_a u^a$. With the wave vector given by (4.1), a simple computation yields $k_a u^a = n/N(1 + \mathcal{O}(\rho))$, giving the standard blue shift near the horizon.

This choice of moding affects the algebra of deformations: because ∂_φ is now $\mathcal{O}(1/\bar{\varepsilon})$, terms that were previously negligible are now important. In particular, it is straightforward to check that

$$\{\tilde{\xi}, \tilde{\eta}\}^t_{SD} = (1+\nu^2)(\tilde{\xi}^t \bar{\partial}_t \tilde{\eta}^t - \tilde{\eta}^t \bar{\partial}_t \tilde{\xi}^t) + \frac{1}{\kappa_H^2}(\bar{\partial}_t \tilde{\xi}^t \bar{\partial}_t{}^2 \tilde{\eta}^t - \bar{\partial}_t \tilde{\eta}^t \bar{\partial}_t{}^2 \tilde{\xi}^t) \ , \tag{4.2}$$

which is no longer a Witt algebra. As noted in Appendix A, though, the full surface deformation bracket must also include terms of the form $\{H[\xi], \eta\}$. With the new moding, these are also no longer negligible. The full brackets (A.14) become

$$\begin{aligned} \{\tilde{\xi}, \tilde{\eta}\}^t_{full} &= (1+\nu^2)(\tilde{\xi}^t \bar{\partial}_t \tilde{\eta}^t - \tilde{\eta}^t \bar{\partial}_t \tilde{\xi}^t) \\ \{\tilde{\xi}, \tilde{\eta}\}^\rho_{full} &= -\rho \bar{\partial}_t \{\tilde{\xi}, \tilde{\eta}\}^t_{full} \\ \{\tilde{\xi}, \tilde{\eta}\}^\varphi_{full} &= \bar{\varepsilon}\nu^2 \{\tilde{\xi}, \tilde{\eta}\}^t_{full} \ , \end{aligned} \tag{4.3}$$

forming a standard Witt algebra and again preserving the relations (3.4).

The central term (3.10) now becomes

$$
\begin{aligned}
K[\xi,\eta] &= -\frac{1}{8\pi G}\int_{\partial\Sigma} d^2x\sqrt{\sigma}\, n^k\Big[(D_i\hat{\xi}_k D^i\eta^\perp - D_i\hat{\xi}^i D_k\eta^\perp) - (\xi\leftrightarrow\eta)\Big] \\
&= -\frac{\nu^2}{4\pi G\kappa_{\scriptscriptstyle H}}\frac{\mathcal{A}}{2\pi}\int d\varphi\ \left[\bar{\partial}_t\xi^t\bar{\partial}_t{}^2\eta^t - \bar{\partial}_t\eta^t\bar{\partial}_t{}^2\xi^t\right]\ ,
\end{aligned}
\tag{4.4}
$$

corresponding to a central charge[‡] of

$$
c = \frac{4\nu^2}{(1+\nu^2)^2}\frac{3\mathcal{A}}{2\pi G\kappa_{\scriptscriptstyle H}} = \frac{3\mathcal{A}}{2\pi G\kappa_{\scriptscriptstyle H}}\ . \tag{4.5}
$$

This again matches the older results of [2,3]. The canonical Cardy formula, with the normal Hawking temperature, then yields

$$
S = \frac{\pi^2}{3}cT = \frac{\mathcal{A}}{4G}\ , \tag{4.6}
$$

reproducing the standard Bekenstein-Hawking entropy. Again, I show in Appendix B that the same result can be obtained from the microcanonical form of the Cardy formula.

It should be noted that although this approach gives a finite temperature and central charge, the behavior is still somewhat singular at the horizon. As observed above, the modes (4.1) are infinitely blue-shifted as the stretched horizon approaches the true horizon. In addition, the ρ derivatives in (3.4) blow up at the horizon. This behavior is also familiar, however: in terms of the usual "tortoise coordinate" $r_*\sim\frac{1}{\kappa}\ln(\kappa\rho)$, we have $\rho\partial_\rho\sim\partial_{r_*}$, so the apparent singular behavior corresponds to the familiar smooth dependence of modes on r_*. The condition (3.6) on ξ^ρ is a bit more problematic, but as noted earlier, this condition is not really needed.

5 Conclusions

The extremal black hole is special: its horizon is an infinite proper distance from any stationary observer, and has an asymptotically anti-de Sitter structure. In retrospect, it is not so surprising that conventional asymptotic methods yield a conformal algebra that describes its states.

To perform a similar analysis in the nonextremal case, one traditionally introduces a stretched horizon and imposes boundary conditions there. We have seen that this procedure is not unique. For the choice that most closely resembles the NHEK approach to the extremal black hole, the entropy appears to be split between the inner and outer horizons, which converge only for the extremal black hole. An alternative choice of "stretching the Killing horizon," on the other hand, yields the full Bekenstein-Hawking entropy at the outer horizon. It has been known for some time that in such an approach, the relevant diffeomorphisms

[‡]If the reciprocal of $\bar{\varepsilon}$ is not an integer, the modes (4.1) are not periodic, and the boundary algebra will receive corrections. These are of order $\bar{\varepsilon}$, though, and are negligible in the horizon limit. Alternatively, one may insist that the stretched horizon be chosen such that $\bar{\varepsilon}=1/M$ for some large integer M.

behave poorly in the horizon limit [27, 28]. But we have now seen that this is at least arguably a physical effect, resulting from the normal infinite blue shift at the horizon.

Ideally, one might hope to clarify these issues by looking at an appropriate algebra of diffeomorphisms at a genuine horizon, without resorting to an intermediate stretched horizon. Unfortunately, the constraint algebra becomes quite complicated on a null surface (see, for instance, [29,30]), with an awkward set of second class constraints. Nevertheless, this avenue seems worth pursuing.

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Appendix A Central terms in Hamiltonian gravity

The symmetries of canonical general relativity are generated by the Hamiltonian

$$H[\xi^\perp,\hat\xi^i] = \int d^3x\left(\xi^\perp\mathcal{H} + \hat\xi^i\mathcal{H}_i\right) \tag{A.1}$$

with

$$\mathcal{H} = \frac{1}{\sqrt{q}}\left(\pi^{ij}\pi_{ij} - \pi^2\right) - \sqrt{q}\,{}^{(3)}\!R, \qquad \mathcal{H}^i = -2D_j\pi^{ij}\ . \tag{A.2}$$

Here, q_{ij} is the spatial metric, π^{ij} is the canonical momentum, D_j is the spatial covariant derivative compatible with q_{ij}, and $\mathcal{H}$ and $\mathcal{H}^i$ are the Hamiltonian and momentum constraints; I use the sign conventions of [31], and units $16\pi G = 1$. On a manifold without boundary, these generators have Poisson brackets

$$\begin{aligned}
\{H[\xi], q_{ij}\} &= -\frac{2}{\sqrt{q}}\xi^\perp\left(\pi_{ij} - \frac{1}{2}q_{ij}\pi\right) - (D_i\hat\xi_j + D_j\hat\xi_i)\\
\left\{H[\xi], \pi^{ij}\right\} &= \sqrt{q}\,\xi^\perp\left({}^{(3)}R^{ij} - \frac{1}{2}q^{ij}\,{}^{(3)}\!R\right) + \frac{2}{\sqrt{q}}\,\xi^\perp\left(\pi^{ik}\pi_k{}^j - \frac{1}{2}\pi\pi^{ij}\right)\\
&\quad - \frac{1}{2}\frac{1}{\sqrt{q}}\,\xi^\perp q^{ij}\left(\pi_{mn}\pi^{mn} - \frac{1}{2}\pi^2\right) - \sqrt{q}\left(D^iD^j\xi^\perp - q^{ij}D_kD^k\xi^\perp\right)\\
&\quad - D_k(\hat\xi^k\pi^{ij}) + \pi^{ik}D_k\hat\xi^j + \pi^{jk}D_k\hat\xi^i
\end{aligned} \tag{A.3}$$

and

$$\{H[\xi], H[\eta]\} = H[\{\xi,\eta\}_{\scriptscriptstyle SD}]\ , \tag{A.4}$$

where the surface deformation brackets [16] are

$$\begin{aligned}
\{\xi,\eta\}^{\perp}_{SD} &= \hat{\xi}^i D_i\eta^\perp - \hat{\eta}^i D_i\xi^\perp \\
\{\xi,\eta\}^{i}_{SD} &= \hat{\xi}^k D_k\hat{\eta}^i - \hat{\eta}^k D_k\hat{\xi}^i + q^{ik}\left(\xi^\perp D_k\eta^\perp - \eta^\perp D_k\xi^\perp\right) \ .
\end{aligned} \tag{A.5}$$

The transformation (A.3) is not a diffeomorphism, but is equivalent on shell to the diffeomorphism generated by a vector field ξ^μ, with

$$\xi^\perp = N\xi^t, \qquad \hat{\xi}^i = \xi^i + N^i\xi^t \tag{A.6}$$

On a manifold with boundary, new complications arise. The generators (A.1), are not "differentiable" [32]: a variation of the fields yields not only the standard functional derivative of the integrand, but also a boundary term, typically singular, from partial integration. One must therefore add a boundary term to $H[\xi]$ to obtain a new generator

$$\bar{H}[\xi] = H[\xi] + B[\xi] \ , \tag{A.7}$$

where the new term $B[\xi]$ depends only on fields and parameters at the boundary, and must be chosen to cancel the boundary terms in the variation of $H[\xi]$.

The specific form of $B[\xi]$ depends on the detailed boundary conditions. Even without knowing these, though, we can say a good deal about the Poisson brackets. By definition, $\bar{H}[\xi]$ has a well-defined variation, with no boundary terms, so

$$\left\{\bar{H}[\xi],\bar{H}[\eta]\right\} = \int d^3x \left(\frac{\delta\bar{H}[\xi]}{\delta q_{ij}}\frac{\delta\bar{H}[\eta]}{\delta\pi^{ij}} - \frac{\delta\bar{H}[\eta]}{\delta q_{ij}}\frac{\delta\bar{H}[\xi]}{\delta\pi^{ij}}\right) \ . \tag{A.8}$$

The functional derivatives in (A.8) can be read off from (A.3). Inserting these and integrating by parts, and denoting the boundary normal and metric by n^i and σ_{ij}, we obtain

$$\left\{\bar{H}[\xi],\bar{H}[\eta]\right\} = \bar{H}[\{\xi,\eta\}_{SD}] + K[\xi,\eta] \ , \tag{A.9}$$

where, restoring factors of G,

$$\begin{aligned}
K[\xi,\eta] = B[\{\xi,\eta\}_{SD}] - \frac{1}{8\pi G}\int_{\partial\Sigma} d^2x\sqrt{\sigma}\, n^k \Big[&\frac{1}{\sqrt{q}}\pi_{ik}\{\xi,\eta\}^i_{SD} - \frac{1}{2}\frac{1}{\sqrt{q}}(\hat{\xi}_k\eta^\perp - \hat{\eta}_k\xi^\perp)\mathcal{H} \\
&+ (D_i\hat{\xi}_k D^i\eta^\perp - D_i\hat{\eta}_k D^i\xi^\perp) - (D_i\hat{\xi}^i D_k\eta^\perp - D_i\hat{\eta}^i D_k\xi^\perp) \\
&+ \frac{1}{\sqrt{q}}\left(\hat{\eta}_k\pi^{mn}D_m\hat{\xi}_n - \hat{\xi}_k\pi^{mn}D_m\hat{\eta}_n\right) + (\xi^\perp\eta^i - \eta^\perp\xi^i)\,{}^{(3)}\!R_{ik}\Big] \ .
\end{aligned} \tag{A.10}$$

This is essentially the same expression as eqn. (4.43) of [33].

To evaluate the central term (A.10), we still need the boundary contribution $B[\{\xi,\eta\}_{SD}]$, which will depend on our particular choice of boundary conditions. For some purposes, though, the details are unnecessary. In particular, suppose we find a Virasoro subalgebra of the group of surface deformations, with $\{\xi,\eta\}_{SD} = \xi\eta' - \eta\xi'$. The boundary term $B[\{\xi,\eta\}_{SD}]$ in (A.10) will then depend only on this combination. A central term in a Virasoro algebra,

on the other hand, looks like $\int d\varphi(\xi'\eta'' - \eta'\xi'')$, a form that cannot be built from $\xi\eta' - \eta\xi'$ and its derivatives. If such a term occurs in $K[\xi,\eta]$, it must thus be a genuine central term.

One final subtlety can sometimes occur in the evaluation of the Poisson brackets (A.4). So far, I have implicitly assumed that the surface deformation parameters $(\xi^\perp, \hat\xi^i)$ are independent of the canonical variables (q_{ij}, π^{ij}). If this is not the case, the Hamiltonian may have nontrivial Poisson brackets with the parameters themselves. In that event, the surface deformation brackets (A.5) become instead

$$\begin{aligned}
\{\xi,\eta\}^\perp_{\mathit{full}} &= \hat\xi^i D_i\eta^\perp - \hat\eta^i D_i\xi^\perp + \{H[\xi],\eta^\perp\} - \{H[\eta],\xi^\perp\}\\
\{\xi,\eta\}^i_{\mathit{full}} &= \hat\xi^k D_k\hat\eta^i - \hat\eta^k D_k\hat\xi^i + q^{ik}\left(\xi^\perp D_k\eta^\perp - \eta^\perp D_k\xi^\perp\right) + \{H[\xi],\hat\eta^i\} - \{H[\eta],\hat\xi^i\}\ .
\end{aligned} \tag{A.11}$$

In particular, in sections 2–4 of this paper, the surface deformation parameters depend on a coordinate ρ which is the proper distance to the horizon on a constant time slice. This proper distance is metric-dependent: in terms of an arbitrary radial coordinate r,

$$\rho(x) = \int_{\mathcal{H}}^x \sqrt{g_{rr}}\,dr\ , \tag{A.12}$$

and hence, from (A.3),

$$\{H[\xi], F(\rho)\} = -\hat\xi^\rho\partial_\rho F + \mathcal{O}(\rho^2)\ . \tag{A.13}$$

(I have omitted a term proportional to the extrinsic curvature K_{rr}, which vanishes at the horizon for the geometries considered here.)

For such metrics, the full surface deformation brackets (A.11) are thus

$$\begin{aligned}
\{\xi,\eta\}^\perp_{\mathit{full}} &= (\hat\xi^\rho\partial_\rho N)\eta^t - (\hat\eta^\rho\partial_\rho N)\xi^t + \hat\xi^\alpha\partial_\alpha\eta^\perp - \hat\eta^\alpha\partial_\alpha\xi^\perp\\
\{\xi,\eta\}^i_{\mathit{full}} &= \hat\xi^\alpha D_\alpha\hat\eta^i - \hat\eta^\alpha D_\alpha\hat\xi^i + (\hat\xi^\rho\partial_\rho N^i)\eta^t - (\hat\eta^\rho\partial_\rho N^i)\xi^t + q^{ik}\left(\xi^\perp D_k\eta^\perp - \eta^\perp D_k\xi^\perp\right)\ ,
\end{aligned} \tag{A.14}$$

where α ranges over the "angular" indices but not ρ. The disappearance of terms of the form $\hat\xi^\rho\partial_\rho\hat\eta^i$ has a simple geometrical explanation: it is simply a consequence of the fact that the proper distance ρ is a diffeomorphism-invariant quantity.

Appendix B Boundary terms near a horizon

As noted in the preceding appendix, the form of the boundary term $B[\xi]$ in the Hamiltonian depends on the particular choice of boundary conditions. While it is not essential to the main line of this paper, it is interesting to work these out at the stretched horizon of a black hole with a metric of the form (2.1)–(2.2).

The boundary terms in the variation of the Hamiltonian (A.1) are [31, 33]

$$\begin{aligned}
\delta H[\xi] = \cdots - \int_{\partial\Sigma} d^2x\Big\{\sqrt{\sigma}\Big[&\xi^\perp(n^k\sigma^{\ell m} - n^m\sigma^{\ell k})D_m\delta q_{k\ell}\\
&- D_m\xi^\perp(n^k\sigma^{\ell m} - n^m\sigma^{\ell k})\delta q_{k\ell}\Big] + 2\hat\xi^i\delta\pi^\rho{}_i - \hat\xi^\rho\pi^{ij}\delta q_{ij}\Big\}\ .
\end{aligned} \tag{B.1}$$

Ordinarily, the boundary metric σ_{ij} is taken to be fixed, and the main contribution to the variation comes from the first term in (B.1). This leads to a boundary term $B[\xi]$ proportional to the extrinsic curvature k of the boundary.

For the black holes investigated here, on the other hand, the extrinsic curvature of the boundary—essentially the expansion—is proportional to ρ, and goes to zero at the horizon. On the other hand, we do not require σ_{ij} to be fixed at the horizon, so the second term in (B.1) gives a nonvanishing contribution. In fact, it is fairly easy to see that for metrics of the form (2.1)–(2.2),

$$\delta H[\xi] = \cdots - \int_{\partial\Sigma} d^2x \left\{ \sqrt{\sigma}\, n^m \partial_m \xi^\perp \sigma^{\ell k} \delta\sigma_{k\ell} - \xi^\perp n^m \partial_m (\sqrt{\sigma}\sigma^{\ell k}\delta\sigma_{k\ell}) + 2\hat{\xi}^i \delta\pi^\rho{}_i \right\} + \mathcal{O}(\rho^2) \ . \tag{B.2}$$

For variations that fix the normal n^a, the required boundary term is thus

$$B[\xi] = \frac{1}{8\pi G}\int_{\mathcal{H}_s} d^2x \left\{ \sqrt{\sigma}\, n^m \partial_m \xi^\perp - \xi^\perp n^m \partial_m \sqrt{\sigma} + \hat{\xi}^i \pi^\rho{}_i \right\} \ , \tag{B.3}$$

where I have restored factors of G. In particular, diffeomorphisms that satisfy condition (3.6)—that is, $\partial_\rho \hat{\xi}^\rho = 0$ at the stretched horizon—leave n^a fixed, and are consistent with this choice of $B[\xi]$. It may further be checked that for deformations of the form (3.4), the three-derivative terms in $\delta_\eta B[\xi]$ match $K[\xi,\eta]$ of eqn. (3.10), as they should [14, 24].

As noted in section 3, though, condition (3.6) is not needed to obtain the central charge. One might worry that without this requirement, our choice of asymptotic symmetries could be inconsistent with the boundary term (B.3). Here, however, we are rescued by a subtlety similar to the one discussed at the end of Appendix A: we must take into account the metric dependence of the proper distance ρ and the coordinate position of the stretched horizon. Explicitly, for a diffeomorphism (3.4) generated by a vector field η^μ, one has

$$\begin{aligned}\delta_\eta \left[\sqrt{\sigma}\, n^m \partial_m \xi^\perp\right] &= (\delta_\eta \sqrt{\sigma})\, n^m \partial_m \xi^\perp - (\partial_\rho \eta^\rho)\sqrt{\sigma}\, n^m \partial_m \xi^\perp \\ &\quad + (\eta^\rho \partial_\rho \sqrt{\sigma})\, n^m \partial_m \xi^\perp + \sqrt{\sigma}\,(\eta^\rho \partial_\rho n^m)\partial_m \xi^\perp + \sqrt{\sigma}\, n^m \partial_m (\eta^\rho \partial_\rho \xi^\perp) \\ &= (\delta_\eta \sqrt{\sigma})\, n^m \partial_m \xi^\perp + \eta^\rho \partial_\rho (\sqrt{\sigma}\, n^m \partial_m \xi^\perp) \ ,\end{aligned} \tag{B.4}$$

where the "extra" $\eta^\rho\partial_\rho$ terms come from the dependence (A.12) of ρ on the metric. The effect of the last term in (B.4) is simply to move the argument of the integrand of (B.3) from ρ to $\rho + \eta^\rho$. But the location of the stretched horizon at $\rho = \rho_s$ is also moved to $\rho + \eta^\rho = \rho_s$, so this produces no change in $B[\xi]$; only the $\delta_\eta\sqrt{\sigma}$ piece remains.

Given the boundary term (B.3), we can now use the microcanonical version of the Cardy formula to check the results of this paper. For the modes (3.8) of section 3,

$$B[\xi_0] = \frac{1}{8\pi G}\int_{\mathcal{H}_s} d^2x\, \sqrt{\sigma} \frac{\kappa_H}{(1+\nu^2)\bar{\varepsilon}} = \frac{\kappa_H \mathcal{A}}{16\pi G \bar{\varepsilon}} \ . \tag{B.5}$$

With the central charge (3.12), the microcanonical Cardy formula then yields

$$S = 2\pi\sqrt{\frac{cL_0}{6}} = \frac{\mathcal{A}}{4G} \ . \tag{B.6}$$

Similarly, for the modes (4.1) of section 4,

$$B[\tilde{\xi}_0] = \frac{1}{8\pi G}\int_{\mathcal{H}_s} d^2x\,\sqrt{\sigma}\frac{\kappa_{_H}}{(1+\nu^2)} = \frac{\kappa_{_H}\mathcal{A}}{16\pi G}\ , \tag{B.7}$$

and the microcanonical Cardy formula with the central charge (4.5) yields

$$S = 2\pi\sqrt{\frac{cL_0}{6}} = \frac{\mathcal{A}}{4G}\ . \tag{B.8}$$

I do not know how to do a corresponding computation for the NHEK-type symmetry of section 2. It is interesting to note, though, that for the extremal case, a choice $L_0 = c/24$ gives the correct entropy (1.15).

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